Math 150, *Lecture Notes- Bonds* **Name**____________

Section 5.3 *Inverse Functions*

Inverse Functions

Recall from Section P.3 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 3$ from $A = \{1, 2, 3, 4\}$ to $B = \{4, 5, 6, 7\}$ can be written as

$$f: \{(1, 4), (2, 5), (3, 6), (4, 7)\}.$$

By interchanging the first and second coordinates of each ordered pair, you can form the **inverse function** of f. This function is denoted by f^{-1}. It is a function from B to A, and can be written as

$$f^{-1}: \{(4, 1), (5, 2), (6, 3), (7, 4)\}.$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 5.10. The functions f and f^{-1} have the effect of "undoing" each other. That is, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x$$

Definition of Inverse Function

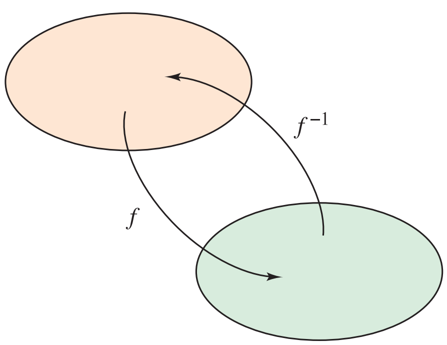
A function g is the **inverse function** of the function f if

$$f(g(x)) = x \quad \text{for each } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \quad \text{for each } x \text{ in the domain of } f.$$

The function g is denoted by f^{-1} (read "f inverse").



Domain of f = range of f^{-1}
Domain of f^{-1} = range of f
Figure 5.10

NOTE Although the notation used to denote an inverse function resembles *exponential notation*, it is a different use of -1 as a superscript. That is, in general, $f^{-1}(x) \neq 1/f(x)$. ■

Here are some important observations about inverse functions.

1. If g is the inverse function of f, then f is the inverse function of g.
2. The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.
3. A function need not have an inverse function, but if it does, the inverse function is unique (see Exercise 108).

You can think of f^{-1} as undoing what has been done by f. For example, subtraction can be used to undo addition, and division can be used to undo multiplication. Use the definition of an inverse function to check the following.

$f(x) = x + c$ and $f^{-1}(x) = x - c$ are inverse functions of each other.

$f(x) = cx$ and $f^{-1}(x) = \dfrac{x}{c}$, $c \neq 0$, are inverse functions of each other.

Ex.1 Verifying Inverse Functions

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

$f(g(x)) = 2\cdot[g(x)]^3 - 1$

$= 2\left[\sqrt[3]{\frac{x+1}{2}}\right]^3 - 1$

$= 2\cdot\left(\frac{x+1}{2}\right) - 1$

$= x + 1 - 1$

$= x$ ✓

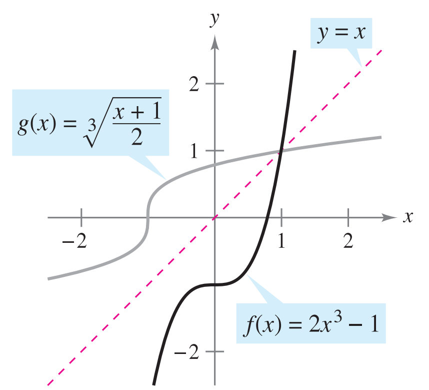
$g(f(x)) = \sqrt[3]{\frac{f(x)+1}{2}}$

$= \sqrt[3]{\frac{(2x^3-1)+1}{2}}$

$= \sqrt[3]{\frac{2x^3}{2}}$

$= \sqrt[3]{x^3}$

$= x$ ✓



f and g are inverse functions of each other.
Figure 5.11

In Figure 5.11, the graphs of f and $g = f^{-1}$ appear to be mirror images of each other with respect to the line $y = x$. The graph of f^{-1} is a **reflection** of the graph of f in the line $y = x$. This idea is generalized in the following theorem.

THEOREM 5.6 Reflective Property of Inverse Functions

The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

PROOF If (a, b) is on the graph of f, then $f(a) = b$ and you can write

$$f^{-1}(b) = f^{-1}(f(a)) = a.$$

So, (b, a) is on the graph of f^{-1}, as shown in Figure 5.12. A similar argument will prove the theorem in the other direction. ■

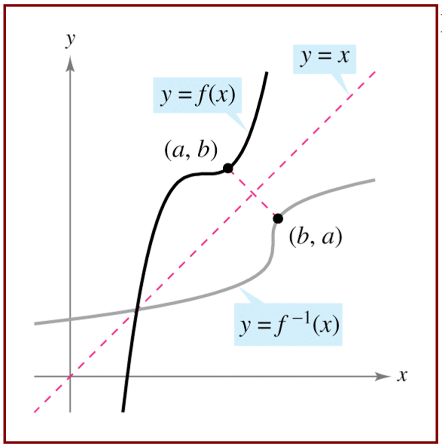


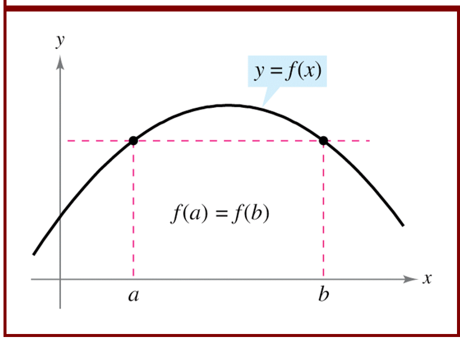
Figure 5.12

Existence of an Inverse Function

Not every function has an inverse function, and Theorem 5.6 suggests a graphical test for those that do—the **Horizontal Line Test** for an inverse function. This test states that a function f has an inverse function if and only if every horizontal line intersects the graph of f at most once (see Figure 5.13). The following theorem formally states why the Horizontal Line Test is valid. (Recall from Section 3.3 that a function is *strictly monotonic* if it is either increasing on its entire domain or decreasing on its entire domain.)

THEOREM 5.7 The Existence of an Inverse Function

1. A function has an inverse function if and only if it is one-to-one.
2. If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.



PROOF To prove the second part of the theorem, recall from Section P.3 that f is one-to-one if for x_1 and x_2 in its domain

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2).$$

Now, choose x_1 and x_2 in the domain of f. If $x_1 \neq x_2$, then, because f is strictly monotonic, it follows that either

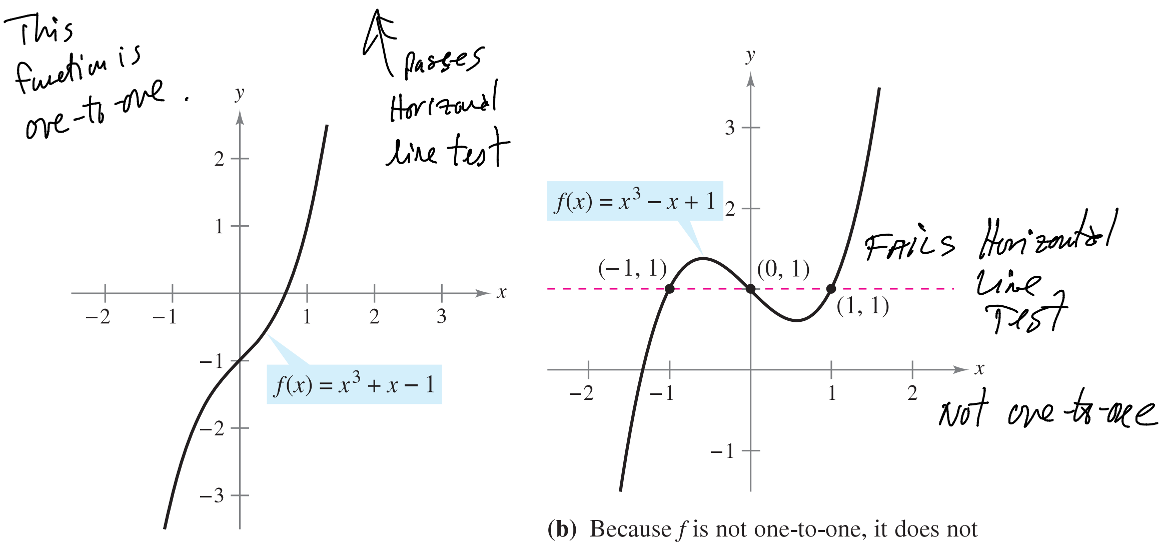
$$f(x_1) < f(x_2) \qquad \text{or} \qquad f(x_1) > f(x_2).$$

In either case, $f(x_1) \neq f(x_2)$. So, f is one-to-one on the interval. The proof of the first part of the theorem is left as an exercise (see Exercise 109). ■

Ex.2 The Existence of an Inverse Function

Which of the functions has an inverse function?

a. $f(x) = x^3 + x - 1$ **b.** $f(x) = x^3 - x + 1$



(a) Because f is increasing over its entire domain, it has an inverse function.

(b) Because f is not one-to-one, it does not have an inverse function.

Figure 5.14

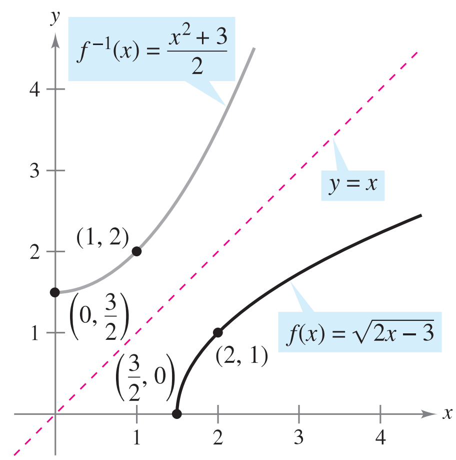
Guidelines for Finding an Inverse Function

1. Use Theorem 5.7 to determine whether the function given by $y = f(x)$ has an inverse function.
2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} to be the range of f.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Ex.3 Finding an Inverse Function

Find the inverse function of

$$f(x) = \sqrt{2x - 3}.$$



The domain of f^{-1}, $[0, \infty)$, is the range of f.
Figure 5.15

Let $y = f(x)$

$y = \sqrt{2x-3}$

trade x & y: $x = \sqrt{2y-3}$, solve for y:

$(x)^2 = [\sqrt{2y-3}]^2$

$x^2 = 2y-3$

$x^2+3 = 2y-3+3$

$x^2+3 = 2y$

$\frac{x^2+3}{2} = \frac{2y}{2}$

$y = \frac{x^2+3}{2}$, Replace $f^{-1}(x) = y$

$f^{-1}(x) = \frac{x^2+3}{2}$

$f(f^{-1}(x)) = \sqrt{2 \cdot f^{-1}(x) - 3}$

$= \sqrt{2 \cdot \left(\frac{x^2+3}{2}\right) - 3}$

$= \sqrt{x^2+3-3}$

$= \sqrt{x^2}$, on $[0, \infty)$

$= x$ ✓

$f^{-1}(f(x)) = \frac{[f(x)]^2+3}{2}$

$= \frac{(\sqrt{2x-3})^2+3}{2}$

$= \frac{2x-3+3}{2}$

$= \frac{2x}{2}$

$= x$ ✓

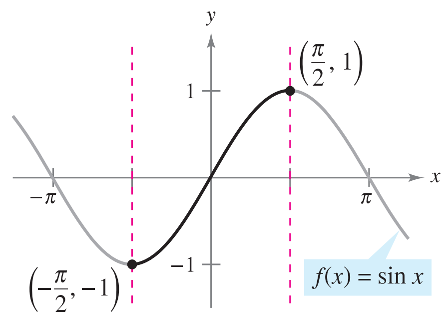
Theorem 5.7 is useful in the following type of problem. Suppose you are given a function that is *not* one-to-one on its domain. By restricting the domain to an interval on which the function is strictly monotonic, you can conclude that the new function *is* one-to-one on the restricted domain.

Ex.4 Testing Whether a Function is One-to-One

Show that the sine function

$$f(x) = \sin x$$

is not one-to-one on the entire real line. Then show that $[-\pi/2, \pi/2]$ is the largest interval, centered at the origin, on which f is strictly monotonic.



f is one-to-one on the interval $[-\pi/2, \pi/2]$.
Figure 5.16

Derivative of an Inverse Function

The next two theorems discuss the derivative of an inverse function. The reasonableness of Theorem 5.8 follows from the reflective property of inverse functions, as shown in Figure 5.12. Proofs of the two theorems are given in Appendix A.

THEOREM 5.8 Continuity and Differentiability of Inverse Functions

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable at c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

THEOREM 5.9 The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

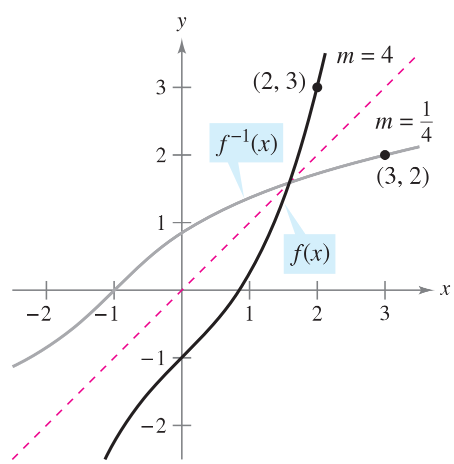
$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

Ex.5 Evaluating the Derivative of an Inverse Function

Let $f(x) = \frac{1}{4}x^3 + x - 1$.

a. What is the value of $f^{-1}(x)$ when $x = 3$?

b. What is the value of $(f^{-1})'(x)$ when $x = 3$?



The graphs of the inverse functions f and f^{-1} have reciprocal slopes at points (a, b) and (b, a).
Figure 5.17

In Example 5, note that at the point $(2, 3)$ the slope of the graph of f is 4 and at the point $(3, 2)$ the slope of the graph of f^{-1} is $\frac{1}{4}$ (see Figure 5.17). This reciprocal relationship (which follows from Theorem 5.9) can be written as shown below. If $y = g(x) = f^{-1}(x)$, then $f(y) = x$ and $f'(y) = \frac{dx}{dy}$. Theorem 5.9 says that

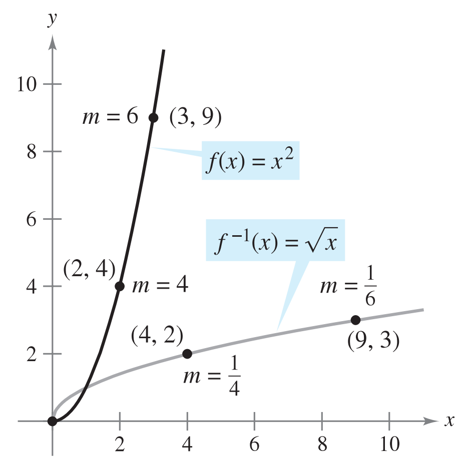
$$g'(x) = \frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{f'(y)} = \frac{1}{(dx/dy)}.$$

So, $$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

Ex.6 Graphs of Inverse Functions have Reciprocal Slopes

Let $f(x) = x^2$ (for $x \geq 0$) and let $f^{-1}(x) = \sqrt{x}$. Show that the slopes of the graphs of f and f^{-1} are reciprocals at each of the following points.

a. $(2, 4)$ and $(4, 2)$ **b.** $(3, 9)$ and $(9, 3)$



At (0, 0), the derivative of f is 0, and the derivative of f^{-1} does not exist.
Figure 5.18